

SOME EXAMPLES IN TORIC GEOMETRY

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ABSTRACT. We present various examples in toric geometry concerning the relationship between smooth toric varieties and quasitoric manifolds (or more generally unitary torus manifolds), and extend the results of [8] to prove the non-existence of almost complex quasitorics over the duals of some certain cyclic 4-polytopes. We also provide the sufficient conditions on the base polytope and the characteristic map so that the resulting quasitoric manifold is almost complex, answering the question proposed by Davis & Januszkiewicz [5] .

1. INTRODUCTION

There is an unfortunate clash on the notion of a *toric manifold*; for instance, a toric manifold for geometers is a smooth toric variety, however, for an algebraic topologist, it is a smooth, even dimensional real manifold acted upon by a compact torus such that the quotient is homeomorphic to a simple convex polytope as a manifold with corners. At first, the class of the latter objects seems to contain the formers (indeed, Davis & Januszkiewicz [5] claimed to do so), we provide an example showing that (see Example 3.2) it is not the case. On the other hand, it is known that any smooth toric variety associated with a fan arising from a polytopal simplicial complex is a toric manifold in the latter sense [4]. Conversely, our Example 3.1 exhibits an almost complex quasitoric manifold that does not arise from a smooth toric variety. To distinguish these two classes, Buchstaber & Panov [2] prefer to call a toric manifold of Davis & Januszkiewicz a *quasitoric manifold*, so do we, and preserve the term *toric manifold* for smooth toric varieties.

The geometric and computational flavor of the theory enables us to translate topological problems into combinatorial ones and vice versa. In this guise, the existence of an almost complex structure compatible with the action on a given quasitoric manifold may be formulated as a combinatorial property carried by the associated simple polytope and the characteristic map, which we characterize in Theorem 2.4 .

In recent years, Masuda's work on unitary torus manifolds [11] contributes to the theory from an unfashionable manner. Instead of starting with some combinatorial ingredients, he begins with a closed, connected, stably complex manifold M^{2n} equipped with a T^n -action such that the T^n -fixed point set is nonempty and isolated, and then he recovers a combinatorial object called *multi-fan* associated to M^{2n} . It can be verified that the class of unitary torus manifolds contains all quasitoric and toric manifolds. However, this containment is strict by Example 5.1. Moreover, under certain restrictions, we can easily adopt our programme in order to characterize the existence of an almost complex structure on unitary torus manifolds (see Corollary 5.2).

Key words and phrases. Quasitoric and toric manifolds, unitary torus manifolds, almost and stably complex structures, cyclic polytopes.

2. ALMOST COMPLEX QUASITORICS

Our first purpose here is to characterize the existence of an almost complex structure on quasitoric manifolds in terms of the related simple polytopes and dicharacteristic maps. So we begin with introducing some notations, and we refer readers to [2] for a more detailed expositions in the theory.

Let M^{2n} be a quasitoric manifold over P^n and let $\mathcal{F} = \{F_1, \dots, F_m\}$ be the set of facets of P^n . Then for each F_i , the pre-image $\pi^{-1}(F_i)$ is a submanifold $M_i^{2(n-1)} \subset M^{2n}$ with isotropy group a circle $T(F_i)$ in T^n . Since there is a one-to-one correspondence (up to a sign) between the set of primitive vectors in \mathbb{Z}^n and the subcircles in T^n , we obtain the *characteristic map* of M^{2n} given by

$$\begin{aligned} \lambda: \mathcal{F} &\rightarrow \mathbb{Z}^n \\ F_i &\mapsto \lambda(F_i) := \lambda_i, \end{aligned}$$

where λ_i generates the circle $T(F_i)$ in T^n . We note that the map λ is well defined only up to a sign, and if the sign of each λ_i is chosen, we then call λ a *dicharacteristic map* of M^{2n} . Therefore, there are 2^m dicharacteristic maps in total attached to M^{2n} . On the other hand, each such choice for λ_i determines an orientation of the normal bundle ν_i of $M_i^{2(n-1)}$, so an orientation for $M_i^{2(n-1)}$. Conversely, an *omniorientation* of M^{2n} consists of a choice of an orientation for every submanifold $M_i^{2(n-1)}$, which in turn settles a sign for each vector λ_i . Thus, every omniorientation is equipped with a unique dicharacteristic map and vice versa. Buchstaber & Ray [3] were able to show that any omniorientation of M^{2n} induces a stably complex structure on it by means of the following isomorphism:

$$(2.1) \quad \tau(M^{2n}) \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \dots \oplus \rho_m,$$

where ρ_i is the pull back of the line bundle corresponding to the Thom class defined by ν_i along the Pontryagin-Thom collapse.

Since P^n is simple, each vertex v of P^n can be written as an intersection of n facets:

$$(2.2) \quad v = F_{i_1} \cap \dots \cap F_{i_n}.$$

Assign to each facet F_{i_k} the edge $E_k : \bigcap_{j \neq k} F_{i_j}$, and let \mathbf{e}_k be the vector along E_k beginning at v . Then, depending on the ordering of the facets (2.2), the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ form either positively or negatively oriented basis of \mathbb{R}^n . Throughout this ordering is assumed to be so that $\mathbf{e}_1, \dots, \mathbf{e}_n$ is a positively oriented basis.

For a given dicharacteristic map λ , we define Λ to be the $(n \times m)$ -matrix whose i -th column is formed by the vector λ_i^t for any $1 \leq i \leq m$. We let $\Lambda_v := \Lambda_{i_1, \dots, i_n}$ denote the maximal minor of Λ formed by the columns i_1, \dots, i_n , where $v = F_{i_1} \cap \dots \cap F_{i_n}$. From the definition of a characteristic map, we have that

$$\det \Lambda_v = \mp 1$$

for any vertex $v \in P^n$.

Definition 2.3. The *sign* of a vertex $v \in P^n$ is defined to be

$$\sigma(v) := \det \Lambda_v$$

Theorem 2.4. An omniorientation of a quasitoric manifold M^{2n} over P^n arises from a T^n -invariant almost complex structure on M^{2n} if and only if $\sigma(v) = 1$ for each vertex $v \in P$.

Proof. The necessity part of the claim has already appeared in [2], so for the sufficiency, assume that $\sigma(v) = 1$ for each vertex $v \in P$. However, this guaranties that the Euler number of the resulting stably complex structure equals to the Euler number of M^{2n} ; hence, it arises from an almost complex structure by the Proposition 4.1 of [13]. It means that the complex structure J on $\tau(M^{2n}) \oplus \mathbb{R}^{2(m-n)}$ splits as $J = (J_1, J_2)$, where J_1 and J_2 are complex structures on $\tau(M^{2n})$ and the trivial portion respectively. However, since J is T^n -invariant, so is J_1 .

On the other hand, the assumption $\sigma(v) = 1$ for each vertex $v \in P$ prevails the fact that each $T(F_i)$ -fixed submanifolds $M_i^{2(n-1)}$ has an almost complex structure induced by that of M^{2n} . This may be achieved systematically by obtaining the dicharacteristic map of $M_i^{2(n-1)}$ from that of M^{2n} such a way that for each fixed point of $M_i^{2(n-1)}$, the sign of the corresponding vertex is equal to 1. \square

3. EXAMPLES

Once we described almost complex structures on quasitorics, it would be of interest to find an example of an almost complex quasitoric that is not a toric manifold, since all known almost complex quasitoric manifolds at the moment are also toric manifolds (see Problem 2.2.11 of [2]).

Example 3.1. Let $M^4 = \mathbb{C}P^2 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ be the quasitoric 4-manifold over the pentagon P oriented counterclockwise shown by the Figure 1 with the given dicharacteristic map (compare to the list given in [[12], p.552], where $\#$ denotes the connected sum. The triangle part of P corresponds to the base polytope of $\mathbb{C}P^2$, and the quadruple portion for $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, where $\overline{\mathbb{C}P^2}$ denotes the complex projective plane with the reversed orientation.

It is easy to check that $\sigma(v_i) = 1$ for any $1 \leq i \leq 5$. Hence, by the Theorem 2.4, the resulting stably complex structure on M^4 is induced by a T^2 -invariant almost complex structure. However, M^4 can not have the diffeomorphic type (or even homeomorphic type) of a toric manifold by the classification theorem of Fischli & Yavin [6]. Therefore, M^4 is an almost complex quasitoric manifold which is not a toric manifold.

We next present a toric manifold that is not a quasitoric manifold.

Example 3.2. Let \mathcal{B}^3 denote the Barnette sphere, which is a star-shaped, non-polytopal simplicial 3-sphere (see [1] or [7]). Since it is star-shaped, it spans a complete fan $\Sigma(\mathcal{B})$ in \mathbb{R}^4 that may not be necessarily smooth. However, applying a finite number of stellar subdivision, we can turn $\Sigma(\mathcal{B})$ into a smooth fan, which we still continue to denote by $\Sigma(\mathcal{B})$. Since the non-polytopality of \mathcal{B}^3 comes from the fact that it has a double edge, the stellar subdivision on \mathcal{B}^3 will not remove such an edge so that the underlying simplicial sphere of $\Sigma(\mathcal{B})$ is still non-polytopal. If we denote the associated toric manifold by $X(\mathcal{B})$, it contains an algebraic torus as a dense subset that acts on $X(\mathcal{B})$ smoothly. Furthermore, the quotient of $X(\mathcal{B})$ by the action of the compact 4-torus included in the algebraic torus is a 4-ball whose facial structure is not isomorphic to a simple convex polytope. Thus, the toric manifold $X(\mathcal{B})$ can not be a quasitoric manifold.

4. QUASITORICS AND CYCLIC POLYTOPES

During the 1990s much of the effort in toric geometry was spent on the classification and the existence of toric manifolds in a purely combinatorial sense. From the geometric point of view, for a given simplicial complex Δ , the existence of a fan $\Sigma(\Delta)$ associated with it depends on the

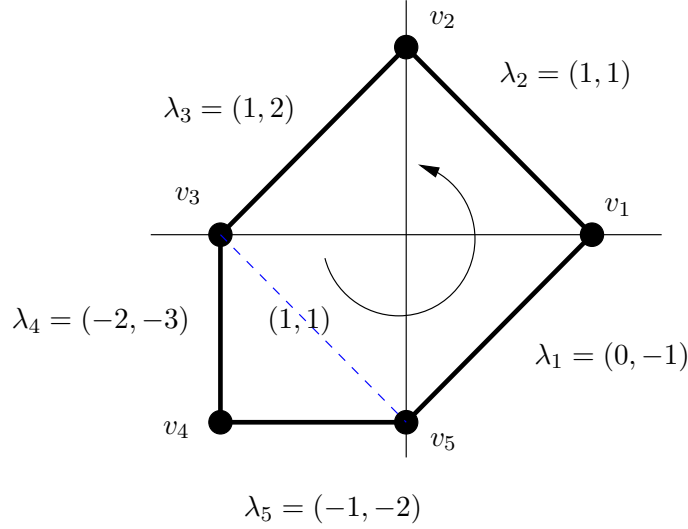


FIGURE 1. The base polytope and a dicharacteristic map for $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$

star-shapness of Δ . However, the smoothness of such a fan can be expressed as a realizability problem in synthetic geometry. In other words, we have to have a realization of $\Sigma(\Delta)$ in such a way that some determinant conditions are satisfied. In the projective case, where the complex Δ is the boundary complex of a simplicial convex polytope, the neighborliness of the polytope puts more restriction on the existence of such realization. In fact, this is the key point on which Gretenkort, et al [8] was able to prove that there does not exist a smooth fan whose spanning simplicial complex is the boundary complex of a cyclic polytope with $n \geq 7$ vertices. Even though, the basic combinatorial ingredients for constructing quasitorics seem to be more flexible, when we require almost complex structures on them, similar synthetic geometry problems will appear.

Instead of using the moment curve to construct a cyclic 4-polytope $C_4(n)$ with n vertices, we may alternatively use the *Carathéodory curve* (see [14]);

$$\mathbf{p}: \mathbb{R} \rightarrow \mathbb{R}^4$$

$$u \mapsto \mathbf{p}(u) := (\cos(u), \sin(u), \cos(2u), \sin(2u))$$

so that $C_4(n) = \text{conv}\{\mathbf{p}(t_1), \dots, \mathbf{p}(t_n)\}$ for $0 \leq t_1 < t_2 < \dots < t_n < 2\pi$. We then denote by $D_4(n)$ the *dual* (or the *polar* when $0 \in \text{relint}(C_4(n))$) of $C_4(n)$. It follows that $D_4(n)$ is a simple convex 4-polytope.

Example 4.1. Consider $0 < \frac{\pi}{4} < \frac{\pi}{2} < \frac{3\pi}{4} < \pi < \frac{5\pi}{4} < \frac{3\pi}{2} < 2\pi$, and let $C_4(7)$ be the cyclic polytope with vertices

$$v_1 = \mathbf{p}(0), v_2 = \mathbf{p}\left(\frac{\pi}{4}\right), v_3 = \mathbf{p}\left(\frac{\pi}{2}\right), v_4 = \mathbf{p}\left(\frac{3\pi}{4}\right),$$

$$v_5 = \mathbf{p}(\pi), v_6 = \mathbf{p}\left(\frac{5\pi}{4}\right), v_7 = \mathbf{p}\left(\frac{3\pi}{2}\right).$$

It is easy to verify that $0 \in \text{relint}(C_4(7))$ so that its polar $D_4(7)$ exists. By the anti-isomorphism between the face lattices of $C_4(7)$ and $D_4(7)$, we denote the facets of $D_4(7)$ by $F_1, F_2, F_3, F_4, F_5, F_6, F_7$ corresponding to the vertices of $C_4(7)$ with the same indexes. By

the Gale's evenness condition and the choice of our realization, the list of positively ordered facets meeting at some vertex of $D_4(7)$ can be given as follows:

1234	1267	2345	3467
2137	3147	2356	4567
2145	4157	2367	
1256	1567	3456	

By analogy with the proof given in [[8], p.257], it can be shown that for any dicharacteristic map on $D_4(7)$, there is at least one vertex v in the above list such that $\sigma(v) = -1$. Therefore, there does not exist a dicharacteristic map on $D_4(7)$ satisfying $\sigma(v) = 1$ for each vertex $v \in D_4(7)$; hence, there is no almost complex quasitoric manifold with the base polytope $D_4(7)$ realized as above. However, we may construct a stably complex quasitoric over $D_4(7)$ with the dicharacteristic map given, for example by

$$\begin{aligned} \lambda(F_1) &= (0, 1, 0, 0), \lambda(F_2) = (1, 0, 0, 0), \lambda(F_3) = (0, 0, 1, 0), \lambda(F_4) = (-1, 0, -1, -1), \\ \lambda(F_5) &= (1, -1, 0, -1), \lambda(F_6) = (1, -1, -1, 0), \lambda(F_7) = (0, 0, 0, 1). \end{aligned}$$

Theorem 4.2. *There does not exist an almost complex quasitoric manifold over the polytope $D_4(n)$ with $n \geq 7$.*

Proof. We first note that the existence of an almost complex structure is independent of any specific geometric realization of the polytope. Therefore, when $n = 7$, the claim follows from the Example 4.1. A similar reason applies to the case $n \geq 8$. \square

5. UNITARY TORUS MANIFOLDS

Since the class of multi-fans contains all convex polytopes as well as fans, the unitary torus manifolds may be thought of a generalization of quasitoric and toric manifolds. In this circumstance, it would be interesting to clarify this containment. In other words, it is not clear whether there exists a unitary torus manifold that is neither a quasitoric nor a toric manifold. We provide such an example in Example 5.1. We may also extend our discussion in Section 2 to deal with the existence of an almost complex structure on unitary torus manifolds under some restrictions. We refer readers to [9] for a more detailed exposition and notation on unitary torus manifolds.

Example 5.1. Let \mathcal{B}^3 denote the Barnette sphere with the following 3-simplicies (see [7]):

$$\begin{aligned} &[x_1, x_2, x_3, x_4] \quad [x_3, x_4, x_5, x_6] \quad [x_1, x_2, x_5, x_6] \quad [x_1, x_2, x_4, x_7] \quad [x_1, x_3, x_4, x_7] \quad [x_3, x_4, x_6, x_7] \\ &[x_3, x_5, x_6, x_7] \quad [x_1, x_2, x_5, x_7] \quad [x_2, x_5, x_6, x_7] \quad [x_2, x_4, x_6, x_7] \quad [x_1, x_2, x_3, x_8] \quad [x_2, x_3, x_4, x_8] \\ &[x_3, x_4, x_5, x_8] \quad [x_4, x_5, x_6, x_8] \quad [x_1, x_2, x_6, x_8] \quad [x_1, x_5, x_6, x_8] \quad [x_1, x_3, x_5, x_8] \quad [x_2, x_4, x_6, x_8] \\ &\text{and } [x_1, x_3, x_5, x_7] \text{ as the base.} \end{aligned}$$

The f and h -vectors of \mathcal{B} are given by $f(\mathcal{B}) = (8, 27, 38, 19)$ and $h(\mathcal{B}) = (1, 4, 9, 4, 1)$. We define $\lambda: V(\mathcal{B}) \rightarrow \mathbb{Z}^4$ by

$$\begin{aligned} \lambda(x_1) &= (1, 0, 0, 0), \quad \lambda(x_2) = (0, 1, -1, 2), \quad \lambda(x_3) = (0, 1, 0, 0), \\ \lambda(x_4) &= (0, 0, 1, -1), \quad \lambda(x_5) = (0, 0, 1, 0), \quad \lambda(x_6) = (1, -1, 0, -1), \\ \lambda(x_7) &= (0, 0, 0, 1), \quad \text{and} \quad \lambda(x_8) = (1, 0, 0, -1), \end{aligned}$$

and denote by Λ , the matrix whose columns consist of vectors $\lambda(x_i)^t$ for $1 \leq i \leq 8$. It then follows that

$$\det \Lambda_\sigma = \mp 1,$$

for each 3-simplex σ of \mathcal{B} , where Λ_σ is the maximal minor of Λ corresponding to the simplex σ . Since \mathcal{B} is a simplicial sphere, the pair (\mathcal{B}, λ) defines a unitary torus manifold $M_\lambda(\mathcal{B})$. It is obvious that \mathcal{B} can not span a smooth fan in \mathbb{R}^4 with the generating set $\{\lambda(x_i) : 1 \leq i \leq 8\}$ so that $M_\lambda(\mathcal{B})$ can not be a toric manifold. Similarly, since \mathcal{B} is a non-polytopal sphere; the manifold $M_\lambda(\mathcal{B})$ is not quasitoric.

We note that any unitary torus 4-manifold is quasitoric by the Steinitz Theorem which asserts that any simplicial 2-sphere is polytopal.

We next determine the condition for a unitary torus manifold in order to carry an almost complex structure under some restrictions. Our limitation here only requires that the Euler characteristic of the manifold equals to the sum of the h -vector of the underlying multi-fan.

Let M^{2n} be a unitary torus manifold such that $M_{I(p)} = p$ for any $p \in M^T$, and assume that if $p = M_{i_1} \cap \dots \cap M_{i_n}$, then $I = \{i_1, \dots, i_n\}$ is a positively oriented simplex in Γ_M . Furthermore, let us set $\sigma(p) = 1$ or -1 according to whether the set $\{v_{i_1}, \dots, v_{i_n}\}$ is a positively or negatively oriented basis of $H_2(BT)$ respectively.

Corollary 5.2. *Under the above assumptions, there exists a T^n -invariant almost complex structure on the unitary torus manifold M^{2n} if and only if $\sigma(p) = 1$ for each $p \in M^T$.*

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